HOMOGENEOUS STRAINS.

By Dr. William H. Metzler, Boston, Mass.

Introduction.

A matrix considered as a linear vector operator, when geometrically interpreted, represents a homogeneous strain,* so that the latter subject may be treated by means of matrices. This subject has been very ably and fully treated by Professors Tait and Kelland in their treatises on Quaternions, and I do not here in any way extend their investigations but simply cover some of the same ground making use of the operator φ in the form of a square array and thus exhibiting the roles which its constituents play.

I shall separate the paper into two parts, the first dealing with homogeneous strains in space of two dimensions, and the second part dealing with homogeneous strains in space of three dimensions.

Part I.—Space of Two Dimensions.

1. Let the matrix $\varphi = \left(\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \right)$,

and let $\rho = xi + yj$ be the vector to a point whose coordinates are (x, y) referred to the rectangular system i, j; then

$$\varphi\rho = (a_{11}x + a_{12}y)\,i + (a_{21}x + a_{22}y)\,j = \rho' .$$

The angle θ between ρ and $\varphi\rho$ is given by

$$\cos\theta = \cos \widehat{\rho \,.\, \varphi\rho} = -\frac{S\rho\varphi\rho}{T\rho \,.\, T\varphi\rho};$$

and therefore

$$\cos\theta = \frac{a_{11}x^2 + xy\,(a_{12} + a_{21}) + y^2 a_{22}}{\sqrt{x^2 + y^2}\,\sqrt{(a_{11}x + a_{12}y)^2 + (a_{21}x + a_{22}y)^2}} .$$

2. If ρ is unchanged in direction by φ we must have $V\rho\varphi\rho = 0$ or $\varphi\rho = \lambda\rho$, from which we get

$$\lambda x = a_{11}x + a_{12}y\,, \quad \lambda y = a_{21}x + a_{22}y\,;$$

or

$$(a_{11} - \lambda)\,x + a_{12}y = 0\,, \quad a_{21}x + (a_{22} - \lambda)\,y = 0\,.$$

* See Taber, American Journal of Mathematics, Vol. XII.

These equations are satisfied by values of x and y other than zero when

$$\Delta' = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0 ;$$

and therefore

$$\lambda = \frac{a_{11} + a_{22} \pm \sqrt{(a_{11} + a_{22})^2 - 4\Delta}}{2} ,$$

where $\Delta = |\varphi| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$.

Denoting these two values of λ, which are the latent roots of, φ by g_1 and g_2 we have

$$\frac{x}{g_1 - a_{22}} = \frac{y}{a_{21}} = m , \quad \text{or} \quad \frac{x}{a_{12}} = \frac{y}{g_1 - a_{11}} = n ;$$

and

$$\frac{x}{g_2 - a_{22}} = \frac{y}{a_{21}} = m , \quad \text{or} \quad \frac{x}{a_{12}} = \frac{y}{g_2 - a_{11}} = n .$$

The two vectors whose directions are unchanged by φ are therefore

$$\rho_1 = m \{ (g_1 - a_{22}) i + a_{21} j \} = n \{ a_{12} i + (g_1 - a_{11}) j \} ,$$

$$\rho_2 = m \{ (g_2 - a_{22}) i + a_{21} j \} = n \{ a_{12} i + (g_2 - a_{11}) j \} .$$

These vectors are real when g_1 and g_2 are real and it is obvious that ρ_1 and ρ_2 will coincide when $g_1 = g_2$.

Operating with φ on ρ_1 and ρ_2, we get

$$\varphi\rho_1 = m \left[\{ a_{11} (g_1 - a_{22}) + a_{12} a_{21} \} i + \{ a_{21} (g_1 - a_{22}) + a_{21} a_{22} \} j \right] = g_1 \rho_1 ;$$

similarly

$$\varphi\rho_2 = g_2 \rho_2$$

Again,

$$\cos \widehat{\rho_1 . \rho_2} = \frac{(g_1 - a_{22})(g_2 - a_{22}) + a_{21}^2}{\sqrt{(g_1 - a_{22})^2 + a_{21}^2} . \sqrt{(g_2 - a_{22})^2 + a_{21}^2}}$$

$$= \frac{a_{21}(a_{12} - a_{21})}{\sqrt{(g_1 - a_{22})^2 + a_{21}^2} . \sqrt{(g_2 - a_{22})^2 + a_{21}^2}}$$

$$= 0 , \text{ if } \varphi \text{ is symmetric.}$$

If φ leaves all vectors unchanged in direction the equation

$$\lambda x = a_{11} x + a_{12} y , \quad \lambda y = a_{21} x + a_{22} y$$

must be independent of x and y, and therefore $a_{12} = a_{21} = 0$ and $a_{11} = a_{22} = \lambda$.

Consequently

$$\varphi = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{11} \end{pmatrix} = a_{11} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and is a multiple of unity.

3. If φ transforms the vector ρ perpendicular to itself we must have $\cos \widehat{\rho . \varphi\rho} = 0$, that is

$$a_{11}x^2 + (a_{12} + a_{21})xy + a_{22}y^2 = 0 ;$$

and $\therefore$

$$\frac{x}{y} = \frac{-a_{12} - a_{21} \pm \sqrt{(a_{12} + a_{21})^2 - 4a_{11}a_{22}}}{2a_{11}}$$

$$= \frac{-a_{12} - a_{21} \pm \sqrt{(a_{12} - a_{21})^2 - 4\Delta}}{2a_{11}} .$$

Then

$$\rho_3 = y\left\{ \left[\frac{-a_{12} - a_{21} + \sqrt{(a_{12} - a_{21})^2 - 4\Delta}}{2a_{11}}\right] i + j \right\} ,$$

$$\rho_4 = y\left\{ \left[\frac{-a_{12} - a_{21} - \sqrt{(a_{12} - a_{21})^2 - 4\Delta}}{2a_{11}}\right] i + j \right\}$$

are two vectors transformed by φ perpendicular to themselves. They are real or imaginary according as $(a_{12} + a_{21})^2 \geqq 4a_{11}a_{22}$ or $(a_{12} + a_{21})^2 < 4a_{11}a_{22}$, respectively.

They are at right angles to each other if $a_{11} + a_{22} = 0$, which is the condition that φ is a vector.*

If φ transforms all vectors perpendicular to themselves then

$$a_{11}x^2 + (a_{12} + a_{21})xy + y^2a_{22} = 0$$

independently of x and y;

$$\therefore \quad a_{11} = a_{22} = 0 , \quad a_{12} = -a_{21} ;$$

and φ is skew symmetric. That is a skew symmetric matrix rotates all vectors through 90° and increases their length a_{12} times.

4. A pure or non-rational strain consists in altering the lengths of two lines at right angles to each other without altering their directions.

We have seen in Art. 2 that the two vectors ρ_1 and ρ_2 are at right angles to each other when φ is symmetric (self-conjugate), which is the condition, therefore, for a pure strain.

5. For a pure rotation,

$$T\varphi\rho \equiv T\rho , \text{ and } \cos \widehat{\rho . \sigma} \equiv \cos \widehat{\varphi\rho . \varphi\sigma} .$$

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* See paper by Prof. Cayley, Messenger of Mathematics, Vol. 14, p. 146.

From either of these we get

$$\sqrt{x^2 + y^2} \equiv \sqrt{(a_{11}x + a_{12}y)^2 + (a_{21}x + a_{22}y)^2}\,;$$

$$\therefore \quad a_{12}^2 + a_{22}^2 = 1\,, \quad a_{11}^2 + a_{21}^2 = 1\,, \quad a_{11}a_{12} + a_{21}a_{22} = 0\,;$$

from which we may derive

$$a_{11}^2 + a_{12}^2 = 1\,, \quad a_{21}^2 + a_{22}^2 = 1\,, \quad a_{11}a_{21} + a_{12}a_{22} = 0\,;$$

and

$$\therefore \quad a_{12} = \pm\, a_{21}\,, \quad a_{11} = \mp\, a_{22}\,,$$

and φ is orthogonal.

Again, $\cos \widehat{\rho \,.\, \varphi\rho} = \dfrac{a_{11}x^2 + xy\,(a_{12} + a_{21}) + a_{22}y^2}{x^2 + y^2}$, which must be the same for all vectors ρ, and consequently independent of x and y;

$$\therefore \quad a_{11} = a_{22}\,, \quad a_{12} = -\,a_{21}\,, \quad \text{and} \quad \cos \widehat{\rho \,.\, \varphi\rho} = a_{11}\,.$$

The direction of rotation is given by the sign of a_{12}; if it is positive the rotation is negative, and if it is negative the rotation is positive.*

We may now write

$$\varphi = \begin{pmatrix} a_{11} & \pm\sqrt{1 - a_{11}^2} \\ \mp\sqrt{1 - a_{11}^2} & a_{11} \end{pmatrix}, \text{ or } \varphi = \begin{pmatrix} -\,a_{11} & \pm\sqrt{1 - a_{11}^2} \\ \mp\sqrt{1 - a_{11}^2} & -\,a_{11} \end{pmatrix}.$$

The latent roots become

$$g_1 = a_{11} + \sqrt{a_{11}^2 - 1}\,, \quad g_2 = a_{11} - \sqrt{a_{11}^2 - 1}\,;$$

both of which are imaginary unless $a_{11}^2 - 1 = 0$, in which case $g_1 = g_2 = \pm 1$. If g_1 and g_2 are imaginary, then obviously ρ_1 and ρ_2 are imaginary; otherwise they are zero.

6. Any matrix φ can be written in the form $\varphi = \varphi_1 + \varphi_2$, where φ_1 is symmetric and φ_2 is skew symmetric; from which we see that any homogeneous strain is equivalent to a pure strain plus a rotation through 90°, accompanied by a uniform dilatation measured by c_{12}, where $c_{12}^2 = |\,\varphi_2\,|$.

Again any matrix φ can be written in the form $\varphi = \varphi_1 + \varphi_2$, where φ_1 is a symmetric and φ_2 is a skew matrix.

Let

$$\varphi_1 = \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix} \quad \text{and} \quad \varphi_2 = \begin{pmatrix} -\,c_{11} & c_{12} \\ -\,c_{12} & c_{11} \end{pmatrix},$$

then $\varphi = \varphi_1 + \varphi_2$ shows that any homogeneous strain is equivalent to a pure

* By a_{12} here is meant the constituent of φ in the first row and second column.

strain plus a rotation through an angle $\cos^{-1}\dfrac{c_{11}}{\sqrt{c_{11}^2 + c_{12}^2}}$, accompanied by a uniform dilatation measured by $\sqrt{c_{11}^2 + c_{12}^2}$.

If $c_{11}^2 + c_{12}^2 = 1$, then φ_2 is orthogonal, and represents a pure rotation. In this case, to express the constituents of φ_1 and φ_2 in terms of those of φ we have

$$a_{11} = b_{11} + c_{11}, \quad a_{12} = b_{12} + \sqrt{1 - c_{11}^2}, \quad a_{21} = b_{12} - \sqrt{1 - c_{11}^2}, \quad a_{22} = b_{22} + c_{11};$$

$$\therefore \quad b_{12} = \tfrac{1}{2}(a_{12} + a_{21}), \quad c_{11} = \pm \tfrac{1}{2}\sqrt{4 - (a_{12} - a_{21})^2},$$

$$b_{11} = a_{11} \mp \tfrac{1}{2}\sqrt{4 - (a_{12} - a_{21})^2}, \quad b_{22} = a_{22} \mp \tfrac{1}{2}\sqrt{4 - (a_{12} - a_{21})^2};$$

and $\therefore$

$$\varphi_1 = \begin{pmatrix} a_{11} \mp \tfrac{1}{2}\sqrt{4 - (a_{12} - a_{21})^2} & \tfrac{1}{2}(a_{12} + a_{21}) \\ \tfrac{1}{2}(a_{12} + a_{21}) & a_{22} \mp \tfrac{1}{2}\sqrt{4 - (a_{12} - a_{21})^2} \end{pmatrix},$$

and

$$\varphi_2 = \begin{pmatrix} \pm \tfrac{1}{2}\sqrt{4 - (a_{12} - a_{21})^2} & \tfrac{1}{2}(a_{12} - a_{21}) \\ -\tfrac{1}{2}(a_{12} - a_{21}) & \pm \tfrac{1}{2}\sqrt{4 - (a_{12} - a_{21})^2} \end{pmatrix};$$

which shows that neither φ_1 nor φ_2 will be real unless $a_{12} - a_{21} \stackrel{=}{<} 2$.

7. If $\varphi = \varphi_1\varphi_2$ where φ_1 is symmetric and φ_2 is orthogonal to express the constituents of φ_1 and φ_2 in terms of those of φ.

From the equation

$$\varphi = \varphi_1\varphi_2$$

we have

$$a_{11} = b_{11}c_{11} - b_{12}\sqrt{1 - c_{11}^2}, \quad a_{12} = b_{11}\sqrt{1 - c_{11}^2} + b_{12}c_{11},$$

$$a_{21} = c_{11}b_{12} - b_{22}\sqrt{1 - c_{11}^2}, \quad a_{22} = b_{12}\sqrt{1 - c_{11}^2} + b_{22}c_{11};$$

$$\therefore \quad c_{11} = \frac{a_{11} + a_{22}}{\sqrt{(a_{11} + a_{22})^2 + (a_{12} - a_{21})^2}},$$

$$\sqrt{1 - c_{11}^2} = \frac{a_{12} - a_{21}}{\sqrt{(a_{11} + a_{22})^2 + (a_{12} - a_{21})^2}},$$

$$b_{11} = \frac{a_{11}(a_{11} + a_{22}) + a_{12}(a_{12} - a_{21})}{\sqrt{(a_{11} + a_{22})^2 + (a_{12} - a_{21})^2}},$$

$$b_{12} = \frac{a_{12}a_{22} + a_{11}a_{21}}{\sqrt{(a_{11} + a_{22})^2 + (a_{12} - a_{21})^2}},$$

$$b_{22} = \frac{a_{22}(a_{11} + a_{22}) - a_{21}(a_{12} - a_{21})}{\sqrt{(a_{11} + a_{22})^2 + (a_{12} - a_{21})^2}}.$$

In this case both φ_1 and φ_2 are real, if φ is real.

8. The matrix $\varphi = \begin{pmatrix} a & a \\ a & a \end{pmatrix}$ rotates all vectors into the direction of $\sigma = i + j$.

If
$$\rho = xi + yj ,$$
then
$$\varphi\rho = a\,(x + y)\,(i + j) = a\,(x + y)\,\sigma ,$$
$$\varphi^2\rho = 2a^2\,(x + y)\,\sigma ,$$
$$\cdots\cdots\cdots$$
$$\varphi^n\rho = 2^{n-1} \,.\, a^n \,.\, (x + y)\,\sigma .$$

PART II.—SPACE OF THREE DIMENSIONS.

9. Let $\varphi = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, and $\rho = xi + yj + zk$;

then
$$\varphi\rho = (a_{11}x + a_{12}y + a_{13}z)\,i + (a_{21}x + a_{22}y + a_{23}z)\,j + (a_{31}x + a_{32}y + a_{33}z)\,k ,$$

and
$$\cos \widehat{\rho \,.\, \varphi\rho} =$$
$$\frac{x^2a_{11} + xy\,(a_{12} + a_{21}) + y^2a_{22} + xz\,(a_{13} + a_{31}) + z^2a_{33} + yz\,(a_{23} + a_{32})}{\sqrt{x^2 + y^2 + z^2}\,\sqrt{(a_{11}x + a_{12}y + a_{13}z)^2 + (a_{21}x + a_{22}y + a_{23}z)^2 + (a_{31}x + a_{32}y + a_{33}z)^2}} .$$

The vectors unchanged by strain are given by $V\rho\varphi\rho = 0$ or $\varphi\rho = \lambda\rho$. From these we get
$$a_{11}x + a_{12}y + a_{13}z = \lambda x ,$$
$$a_{21}x + a_{22}y + a_{23}z = \lambda y ,$$
$$a_{31}x + a_{32}y + a_{33}z = \lambda z .$$

There are three values of λ, given by $\Delta' = 0$, for which these equations are satisfied by values of x, y, and z other than zero, where
$$\Delta' = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix} .$$

Then

$$\frac{x}{A'_{11}} = \frac{y}{A'_{12}} = \frac{z}{A'_{13}} = l\,,$$

$$\frac{x}{A'_{21}} = \frac{y}{A'_{22}} = \frac{z}{A'_{23}} = m\,;$$

or

$$\frac{x}{A'_{31}} = \frac{y}{A'_{32}} = \frac{z}{A'_{33}} = n\,;$$

where

$$A'_{11} = \begin{vmatrix} a_{22} - \lambda & a_{23} \\ a_{32} & a_{33} - \lambda \end{vmatrix}, \quad A'_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} - \lambda \end{vmatrix}, \text{ etc.}$$

Denoting the latent roots of φ by g_1, g_2, g_3, we have

$$\begin{aligned}\rho_1 &= l\,[\{g_1^2 - g_1(a_{22} + a_{33}) + A_{11}\}\,i + \{g_1 a_{21} + A_{12}\}\,j + \{g_1 a_{31} + A_{13}\}\,k] \\ &= m\,[\{g_1 a_{12} + A_{21}\}\,i + \{g_1^2 - g_1(a_{11} + a_{33}) + A_{22}\}\,j + \{g_1 a_{32} + A_{23}\}\,k] \\ &= n\,[\{g_1 a_{13} + A_{31}\}\,i + \{g_1 a_{23} + A_{32}\}\,j + \{g_1^2 - g_1(a_{11} + a_{22}) + A_{33}\}\,k]\,,\end{aligned}$$

$$\rho_2 = l\,[\{g_2^2 - g_2(a_{22} + a_{33}) + A_{11}\}\,i + \{g_2 a_{21} + A_{12}\}\,j + \{g_2 a_{31} + A_{13}\}\,k] = \text{etc.},$$

$$\rho_3 = l\,[\{g_3^2 - g_3(a_{22} + a_{33}) + A_{11}\}\,i + \{g_3 a_{21} + A_{12}\}\,j + \{g_3 a_{31} + A_{13}\}\,k] = \text{etc.}$$

There are therefore three vectors whose directions are unchanged by the strain, all of which are real if g_1, g_2, and g_3 are real. One at least of these vectors is real, since one of the latent roots must be real.

Operating on ρ_1, ρ_2, ρ_3 by φ, we get

$$\varphi\rho_1 = g_1\rho_1\,, \quad \varphi\rho_2 = g_2\rho_2\,, \quad \varphi\rho_3 = g_3\rho_3\,.$$

10. If φ is symmetric, then g_1, g_2, g_3 are real, ρ_1, ρ_2, ρ_3 form a rectangular system, and the strain is pure. The equations $\varphi\rho_1 = g_1\rho_1$, $\varphi\rho_2 = g_2\rho_2$, $\varphi\rho_3 = g_3\rho_3$ show that for true physical pure strain g_1, g_2, g_3, besides being real, must be positive.*

11. For a pure rotation we must have

$$T\varphi\rho = T\rho\,, \quad \cos\widehat{\rho\,.\,\sigma} = \cos\widehat{\varphi\rho\,.\,\varphi\sigma}\,;$$

$$\begin{aligned} \therefore\quad a_{11}^2 + a_{21}^2 + a_{31}^2 &= 1\,, & a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} &= 0\,, \\ a_{12}^2 + a_{22}^2 + a_{32}^2 &= 1\,, & a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33} &= 0\,, \\ a_{13}^2 + a_{23}^2 + a_{33}^2 &= 1\,, & a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} &= 0\,. \end{aligned}$$

* Kelland and Tait, 2d Ed., Chap. X, Sect. V.

From these we may obtain

$$a_{11}^2 + a_{12}^2 + a_{13}^2 = 1\,, \qquad a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} = 0\,,$$

$$a_{21}^2 + a_{22}^2 + a_{23}^2 = 1\,, \qquad a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33} = 0\,,$$

$$a_{31}^2 + a_{32}^2 + a_{33}^2 = 1\,, \qquad a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} = 0\,.$$

The matrix φ is then orthogonal and $\therefore a_{11} = A_{11}$, and in general, $a_{rs} = A_{rs}$.*

One of the latent roots is real and equal to ± 1,* suppose it to be g_1. Then

$$\rho_1 = l\{(2 \pm (g_2 + g_3 + a_{11}) + A_{11})i + (A_{12} \pm a_{21})j + (A_{13} \pm a_{31})k\}.$$

If all the roots are real, then φ is symmetric, and $g_2 = \pm 1$, $g_3 = \pm 1$.* If $g_1 = g_2 = g_3 = 1$, there is obviously no rotation, and if $g_1 = g_2 = g_3 = -1$, then the rotation is improper or physically impossible.

To find the amount of rotation, we find a vector ρ_4 perpendicular to ρ_1, the axis of rotation, and then the angle θ of rotation is given by

$$\cos\theta = \cos \widehat{\rho_4 \cdot \varphi\rho_4}\,.$$

The vector $\rho_4 = (a_{11} - 1)i + a_{12}j + a_{13}k$ is easily seen to be perpendicular to ρ_1.

Then
$$\varphi\rho_4 = (1 - a_{11})i - a_{21}j - a_{31}k\,,$$
and

$$\begin{aligned}
\cos\theta &= \frac{-(a_{11}-1)^2 - a_{21}a_{12} - a_{13}a_{31}}{\sqrt{(a_{11}-1)^2 + a_{12}^2 + a_{13}^2}\,\sqrt{(a_{11}-1)^2 + a_{21}^2 + a_{31}^2}} \\
&= \tfrac{1}{2}(-a_{11}^2 + 2a_{11} - 1 - a_{11}a_{33} - a_{11}a_{22} + A_{33} + A_{22})(1 - a_{11})^{-1} \\
&= \tfrac{1}{2}[(a_{11} + a_{22} + a_{33}) - 1] \\
&= \tfrac{1}{2}(g_1 + g_2 + g_3 - 1)\,,\dagger \\
\cos 2\theta &= \tfrac{1}{2}(g_1^2 + g_2^2 + g_3^2 - 1)\,,
\end{aligned}$$

and generally

$$\cos n\theta = \tfrac{1}{2}(g_1^n + g_2^n + g_3^n - 1)\,.$$

12. Suppose φ is a skew matrix of the form

$$\varphi = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ -a_{12} & a_{11} & a_{23} \\ -a_{13} & -a_{23} & a_{11} \end{pmatrix}.$$

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* See my paper, American Journal of Mathematics, Vol. XV, No. 3, § 4.

† See Routh, Rigid Dynamics, 3d Ed., Chap. V, Art. 184.

One of the latent roots (g_1 say) will be real and equal to a_{11}, while the remaining two will be complex imaginary. The vectors ρ_2 and ρ_3 will be imaginary while $\rho_1 = la_{23}(a_{23}i - a_{13}j + a_{12}k)$ and $\varphi\rho_1 = a_{11}\rho_1$.

The vector $\rho_4 = (a^{13} - a_{12})i - (a_{12} - a_{23})j + (a_{23} - a_{13})k$ is readily seen to be perpendicular to ρ_1.

Then

$$\begin{aligned}\varphi\rho_4 = &\{a_{11}(a_{13} - a_{12}) - a_{12}(a_{12} - a_{23}) + a_{13}(a_{23} - a_{13})\}\, i \\ &+ \{-a_{12}(a_{13} - a_{12}) - a_{11}(a_{12} - a_{23}) + a_{23}(a_{23} - a_{13})\}\, j \\ &+ \{-a_{13}(a_{13} - a_{12}) + a_{23}(a_{12} - a_{23}) + a_{11}(a_{23} - a_{13})\}\, k\,,\end{aligned}$$

and

$$\cos \widehat{\rho_4 \cdot \varphi\rho_4} = \frac{a_{11}}{\sqrt{a_{11}^2 + a_{12}^2 + a_{13}^2 + a_{23}^2}};$$

which is seen to be the same for all vectors perpendicular to ρ_1.

We also have

$$T\varphi\rho_4 = \sqrt{a_{11}^2 + a_{12}^2 + a_{13}^2 + a_{23}^2} \,.\, T\rho_4\,,$$

which shows that there is a uniform dilatation perpendicular to the axis of rotation measured by $\sqrt{a_{11}^2 + a_{12}^2 + a_{13}^2 + a_{23}^2}$.

13. If φ is skew symmetric, then $a_{11} = 0$, and therefore $\cos \widehat{\rho_4 \cdot \varphi\rho_4} = 0$; that is, φ rotates all vectors perpendicular to ρ_1 through 90°, and uniformly elongates them $\sqrt{a_{12}^2 + a_{13}^2 + a_{23}^2}$ times.

Postscript.—On obtaining the above results for a skew symmetric matrix I observed that they were at variance with those given in Tait's Quarternions, 3d Ed., Chap. XI, Art. 381, pp. 298. I sent my solution to Prof. Tait, who replied that the fallacy was in his book, and not in my work. He was evidently thinking of another case when giving the results for this.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY, *January* 13, 1894.